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On a time-splitting method for stochastic scalar conservation laws with the initial-boundary condition*

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1 Introduction

We consider the stochastic conservation laws of the following type

$$du + \mathrm{div}(A(u))dt = \Phi(u)dW(t) \quad \text{in} \quad \Omega \times Q, \tag{1.1}$$

with the initial condition

$$u(\cdot, 0) = u_0(\cdot) \quad \text{in} \quad \Omega \times D, \tag{1.2}$$

and the formal boundary condition

$$\text{“}u = u_b\text{”} \quad \text{on} \quad \Omega \times \Sigma. \tag{1.3}$$

Here $D \subset \mathbb{R}^d$ is a bounded convex domain with a Lipschitz boundary ∂D, $T > 0$, $Q = D \times (0, T)$, $\Sigma = \partial D \times (0, T)$ and W is a cylindrical Wiener process defined on a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t), P)$.

In the deterministic case ($\Phi = 0$), the problem has been extensively studied. It is well-known that a smooth solution is constant along characteristic curves, which can intersect each other and shocks can occur. Moreover, when the characteristic intersects both $\{0\} \times D$ and Σ, the problem (1.1)-(1.3) would be overdetermined if (1.3) were assumed in the usual sense. Thus, an

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appropriate framework of entropy solutions, together with entropy-boundary conditions, has been considered to obtain the well-posedness of (1.1)-(1.3) with $\Phi = 0$. Bardos, Le Roux and Nédélec [2] first gave an interpretation of the boundary condition (1.3) as an entropy inequality on Σ. However, their result requires the existence of trace on Σ with respect to L^1 strong topology, and so they had to consider solutions in the BV setting. Otto [17] has extended their result to the L^∞ setting by introducing the notion of boundary entropy flux pairs. On the other hand, Imbert and Vovelle [11] gave a kinetic formulation to (1.1)-(1.3) with $\Phi = 0$ and proved the uniqueness of kinetic solutions in the L^∞ space. Concerning the Cauchy-Dirichlet problem for deterministic degenerate parabolic equations, see [19, 13].

As regard the Cauchy problem for the stochastic case it has been studied in [12] in the case of additive noise, in [8] in the case of multiplicative noise, where the uniqueness of the "strong" entropy solution is established in any dimension, but the existence in one dimension. For the existence in any dimension see [3]. The Cauchy problem for (1.1) with a multiplicative noise $\Phi(u)dW(t)$ in a d-dimensional torus has been studied in [5], in which Debussche and Vovelle proved the well-posedness of (1.1) by using a kinetic formulation. The main advantage in using kinetic formulations developed by Lions, Perthame and Tadmor [18] is that the formulation keeps track of the dissipation of noise by solutions. Those results have been extended to the case of degenerated parabolic stochastic equations in [4, 15].

There are several papers concerning the Cauchy-Dirichlet problem for stochastic conservation laws. Vallet and Wittobold [21] extended the result of Kim [12] to the d-dimensional Cauchy-Dirichlet problem with additive noise, and then Bauzet, Vallet and Wittbold [1] studied in the case of multiplicative noise. In [21, 1] it is assumed that the flux A is global Lipschitz and the Dirichlet boundary datum is zero. The homogeneous boundary condition is formulated in the sense of Carrillo, which formulates the semi-Kružkov entropies.

In the recent paper [14] Kobayasi and Noboriguchi investigated the non-homogeneous Dirichlet boundary problem (1.1)-(1.3) under the hypothesis (H_1)-(H_3). The hypothesis (H_1) implies that the flux A is not always Lipschitz but locally Lipshitz, and hence an important example of inviscid Burgers' equation can be included. The basic idea of the arguments in [14] is analogous to that of [5, 11], but the stochastic case is significantly different from the deterministic case. A "stochastic kinetic solution" u might blow up at the boundary ∂D even if the data u_0, u_b are bounded. The defect measure $\bar{m}^{\pm}$

on the boundary $\Sigma \times \mathbb{R}_\xi$ play an important role. In particular, it is crucial that $\bar{m}^+$ (resp. $\bar{m}^-$) vanishes for $\xi >> 1$ (resp. $\xi << -1$) in the proof of uniqueness. These properties for $\bar{m}^+$, $\bar{m}^-$ come from the boundedness of kinetic solutions. However, in the stochastic case we have no pathwise L^∞ estimate of kinetic solution u even though the data u_0, u_b belong to L^∞: It is known only that $\mathbb{E}\sup_{0\le t\le T}\|u(t)\|^p_{L^p(D)}$ is finite for every $p \in [1,\infty)$ and hence we are not able to obtain that the boundary defect measures $\bar{m}^+$, $\bar{m}^-$ vanish as $\xi \to \infty$, $\xi \to -\infty$. To overcome this difficulty the notion of "renormalized" kinetic formulations (see (2.3) below) is introduced in [14], in which $\bar{m}^+$, $\bar{m}^-$ are cut off on each finite interval $(-N, N)$ of $\mathbb{R}_\xi$. By renormalizing the kinetic formulation we proved in [14] the uniqueness of such a solution. However, in order to obtain the existence we needed to add several thechnical assumptions on the flux A and data u_0, u_b. This need is due to the fact that the existence is obtained by approximating the equation (1.1) by appropriate stochastic parabolic equations which are solvable by the result of [9].

The purpose of the present article is to give a summary of the original paper [16] in which we established the existence of the kinetic solution to (1.1)-(1.3) by assuming the hypothesis (H_1)-(H_3) only. We proved it by a time-splitting method. To be more precise, let $\mathcal{R}(t,s)v_s$ denote the solution of the purely stochastic equation (3.1) below with the initial datum v_s at $t = s$, and let $\mathcal{S}(t-s)w_s$ denote the solution of the deterministic conservation law (3.2) with the initial datum w_s at $t = s$ and the boundary datum u_b on $(s,T)\times\partial D$. Given $\varepsilon > 0$ let $0 = t_0^\varepsilon < t_1^\varepsilon < \cdots < t_{N_\varepsilon}^\varepsilon = T$ be a partition of the interval $[0,T]$ such that the mesh size tends to 0 as $\varepsilon \to 0$. Consider the type of Lie-Trotter's product formula:

$$v^\varepsilon(t) = \mathcal{R}(t,t_n^\varepsilon)\prod_{k=1}^{n}[\mathcal{S}(t_k^\varepsilon - t_{k-1}^\varepsilon)\mathcal{R}(t_k^\varepsilon, t_{k-1}^\varepsilon)]u_0$$

for $t \in [0,T)$ where n is the integer such that $t \in [t_n^\varepsilon, t_{n+1}^\varepsilon)$. Then $v^\varepsilon(x,t)$ converges in the L^1 sense to a kinetic solution of (1.1)-(1.3) as $\varepsilon \to 0$. In order to discuss this convergence in the L^1 setting we need to choose an appropriate partition $\{t_n^\varepsilon\}$ of $[0,T]$.

We now give the precise hypothesis in this article:

(H_1) The flux function A is of class $C^2(\mathbb{R};\mathbb{R}^d)$ and its derivative denoted by $a = (a_1,\ldots,a_d)$ have at most polynomial growth.

(H$_2$) For each $z \in L^2(D)$, $\Phi(z) : H \to L^2(D)$ is defined by $\Phi(z)e_k = g_k(\cdot, z(\cdot))$, where $g_k \in C(D \times \mathbb{R})$ satisfies the following conditions:

$$G^2(x,\xi) = \sum_{k=1}^{\infty} |g_k(x,\xi)|^2 \leq C(1+|\xi|^2), \tag{1.4}$$

$$\sum_{k=1}^{\infty} |g_k(x,\xi) - g_k(y,\zeta)|^2 \leq C\Big(|x-y|^2 + |\xi-\zeta| r(|\xi-\zeta|)\Big) \tag{1.5}$$

for every $x, y \in D$, $\xi, \zeta \in \mathbb{R}$. Here C is a constant and r is a continuous non-decreasing function on $\mathbb{R}_+$ with $r(0) = 0$.

(H$_3$) $u_0 \in L^p(\Omega, \mathscr{F}_0; L^p(D))$ for all $p \in [1, \infty)$ and $u_b \in L^\infty(\partial D \times (0,T))$.

This article is organized as follows. In Section 2 we introduce the notion of kinetic solutions to (1.1)-(1.3) by using the renormalized kinetic formulation and state the main result of the well-posedness. In Section 3 we construct approximate solutions to (1.1)-(1.3) and give some fundamental lemmas concerning these approximations. In Section 4 we give an outline of the proof of the existence part of the main theorem.

2 The main result

We give the definition of solution and the main result in this section.

Definition 2.1 (Kinetic measure). *A map m from Ω to $\mathcal{M}_b^+(D \times [0,T) \times \mathbb{R})$, the set of non-negative finite measures over $D \times [0,T) \times \mathbb{R}$, is said to be a kinetic measure if*

(i) *m is weakly measurable, i.e., for each $\phi \in C_b(D \times [0,T) \times \mathbb{R})$ the map $m(\phi) : \Omega \to \mathbb{R}$ is measurable,*

(ii) *m vanishes for large ξ in the following sense:*

$$\lim_{R \to \infty} \mathbb{E} m(D \times [0,T) \times \{\xi \in \mathbb{R}; |\xi| \geq R\}) = 0, \tag{2.1}$$

(iii) *for all $\phi \in C_b(D \times \mathbb{R})$, the process*

$$t \mapsto \int_{D \times [0,t] \times \mathbb{R}} \phi(x,\xi)\, dm(x,s,\xi) \tag{2.2}$$

is predictable.

In order to define kinetic solutions, we now introduce equilibrium functions $f^{\pm}$ defined by

$$f^{+}(u,\xi)=\begin{cases}1 & \text{if } \xi<u,\\ 0 & \text{if } \xi\geq u,\end{cases} \quad \text{and} \quad f^{-}(u,\xi)=\begin{cases}-1 & \text{if } \xi>u,\\ 0 & \text{if } \xi\leq u.\end{cases}$$

Then, kinetic solutions are defined as follows.

Definition 2.2 (Kinetic solution). *Let $u_0\in L^p(\Omega,\mathscr{F}_0;L^p(D))$ for all $p\in[1,\infty)$, $u_b\in L^\infty(\Sigma)$ and let $u\in L^p(\Omega\times[0,T),\mathcal{P};L^p(D))\cap L^p(\Omega;L^\infty(0,T;L^p(D)))$ for all $p\in[1,\infty)$, where $\mathcal{P}$ is the predictable σ-algebra on $\Omega\times[0,T)$ associated to $(\mathscr{F}_t)$. Then u is said to be a kinetic solution to (1.1)-(1.3) if there exists a kinetic measure m and for any $R>0$ there exist non-negative $\bar{m}_R^{\pm}\in L^1(\Omega\times\Sigma\times(-R,R))$ such that $\{\bar{m}_R^{\pm}(t)\}$ are predictable, $\bar{m}_R^{+}(R-0)=\bar{m}_R^{-}(-R+0)=0$ for sufficiently large R and u satisfies a kinetic formulation: for all $\varphi\in C_c^\infty(\overline{D}\times[0,T)\times\mathbb{R})$ with $\varphi(x,t,\xi)=0$, $|\xi|\geq R$,*

$$\begin{aligned}
&\int_{Q\times\mathbb{R}} f^{\pm}(u,\xi)(\partial_t+a(\xi)\cdot\nabla)\varphi\; d\xi dxdt\\
&\quad+\int_{D\times\mathbb{R}} f^{\pm}(u_0,\xi)\varphi(0)\; d\xi dx+M_R\int_{\Sigma\times\mathbb{R}} f^{\pm}(u_b,\xi)\varphi\; d\xi d\sigma dt\\
&=-\sum_{k=1}^{\infty}\int_0^T\int_D g_k(x,u)\varphi(x,t,u)\; dxd\beta_k(t)\\
&\quad-\frac{1}{2}\int_Q G^2(x,u)\partial_\xi\varphi(x,t,u)\; dxdt\\
&\quad+\int_{D\times[0,T)\times\mathbb{R}}\partial_\xi\varphi\; dm+\int_{\Sigma\times\mathbb{R}}\bar{m}_R^{\pm}\partial_\xi\varphi\; d\xi d\sigma dt,\quad a.s.,
\end{aligned}\tag{2.3}$$

where $M_R=\max_{|\xi|\leq R}|a(\xi)|$.

We are now in a position to state our main result.

Theorem 2.3. *Let D be a convex and bounded domain of $\mathbb{R}^d$ with a Lipschitz boundary. Under the assumptions (H_1)-(H_3), there exists a unique kinetic solution to (1.1)-(1.3), which has almost surely continuous orbits in $L^p(D)$. Moreover, for all $t\in[0,T)$,*

$$\begin{aligned}
&\mathbb{E}\|u_1(t)-u_2(t)\|_{L^1(D)}\\
&\quad\leq\mathbb{E}\|u_{1,0}-u_{2,0}\|_{L^1(D)}+M_b\int_0^t\|u_{1,b}(s)-u_{2,b}(s)\|_{L^1(\partial D)}\; ds
\end{aligned}$$

where $M_b = \max\{|a(\xi)| : |\xi| \leq \max_{i=1,2} \|u_{i,b}\|_{L^\infty(\Sigma)}\}$ *and* u_i, $i = 1, 2$, *are kinetic solutions to (1.1)-(1.3) with data* $(u_{i,0}, u_{i,b})$, *respectively.*

3 Construction of approximate solutions

Let us now explain the construction and some properties of the approximate solutions. We consider the following two equations: for $0 \leq s < T$,

$$\begin{cases} dv = \Phi(v)dW(t) & \text{in } D \times (s,T) \\ v(\cdot, s) = v_s(\cdot) & \text{in } D, \end{cases} \tag{3.1}$$

and

$$\begin{cases} \partial_t w + \operatorname{div}(A(w)) = 0 & \text{in } D \times (s,T) \\ w(\cdot, s) = w_s(\cdot) & \text{in } D \\ w \cong u_b & \text{on } \partial D \times (s,T). \end{cases} \tag{3.2}$$

Let $\mathcal{R}(t,s)$ and $\mathcal{S}(t-s)$ be the solution operators of (3.1) and (3.2), respectively. Namely we can write

$$v(t,s) = \mathcal{R}(t,s)v_s \quad \text{and} \quad w(t,s) = \mathcal{S}(t-s)w_s.$$

For the SDE (3.1) we have

Proposition 3.1. *Let* $v_s \in L^p(\Omega; \mathscr{F}_s, dP; L^p(D))$ *for* $p \geq 1$. *There exists a unique kinetic solution* $v(t,s)$ *to (3.1), which has a representative in* $L^p(\Omega; L^\infty(s,T; L^p(D)))$ *with almost surely continuous trajectories in* $L^p(D)$. *Besides, it satisfies the following "strong" kinetic formulation at all* $t \in [s,T)$, *that is, weak in* (x,ξ) *only: P-a.s., for all* $t \in [s,T)$, *for all* $\varphi \in C_c^\infty(D \times \mathbb{R})$,

$$\begin{aligned} &- \int_D \int_{\mathbb{R}} f^{\pm}(v(t,s),\xi)\varphi \, d\xi dx + \int_D \int_{\mathbb{R}} f^{\pm}(v_s,\xi)\varphi \, d\xi dx \\ &= -\sum_{k=1}^{\infty} \int_s^t \int_D g_k(x, v(r,s))\varphi(x, v(r,s)) \, dx d\beta_k(r) \\ &\quad - \frac{1}{2} \int_s^t \int_D G^2(x, v(r,s)) \partial_\xi \varphi(x, v(r,s)) \, dx dr. \end{aligned} \tag{3.3}$$

Moreover, for any $p \geq 2$ *there exists a constant* $C_p \geq 0$ *such that*

$$\mathbb{E} \sup_{t \in [0,T)} \|v(t)\|_{L^p(D)}^p \leq C_p. \tag{3.4}$$

On the other hand, we have the well-posedness of the deterministic scalar conservation law (3.2).

Proposition 3.2. *Let $w_s \in L^p(D)$ for $p \geq 1$. There exists a unique kinetic solution $w(t,s) \in C([s,T); L^1(D))$ of (3.2) which is defined by Definition 2.2 with $\Phi \equiv 0$. Besides we have for all $t \in [s,T)$, $R \geq \|w_{1,b}\|_{L^\infty(\Sigma)}$ and $p \in [1,\infty]$,*

$$\begin{aligned}\|w_1(t) - w_2(t)\|_{L^1(D)}& \\ \leq \|w_1(s) - w_2(s)\|_{L^1(D)} &+ M_b \int_0^t \|w_{1,b}(s) - w_{2,b}(s)\|_{L^1(D)} ds\end{aligned} \tag{3.5}$$

and

$$\|(w_1(t) \mp R)^\pm\|_{L^p(D)} \leq \|(w_1(s) \mp R)^\pm\|_{L^p(D)}, \tag{3.6}$$

where $M_b = \max\{|a(\xi)| : |\xi| \leq \max_{i=1,2} \|w_{i,b}\|_{L^\infty(\Sigma)}\}$, w_i, $i = 1,2$, are arbitrary kinetic solutions to (3.2) with data $(w_{i,0}, w_{i,b})$, respectively.

To prove the existence result we propose to approximate the equations (1.1)-(1.3) as follows. Let $\varepsilon > 0$ and let $t_0^\varepsilon = 0$, $\tilde{u}_0^\varepsilon = u_0$. For $n \in \mathbb{N} \cup \{0\}$, if $t_n^\varepsilon < T$, define

$$\begin{aligned}&t_{n+1}^\varepsilon := \inf\{t > t_n^\varepsilon;\ \mathbb{E}\|\mathcal{S}(t - t_n^\varepsilon)\tilde{u}_n^\varepsilon - \tilde{u}_n^\varepsilon\|_{L^1(D)} > \varepsilon\} \wedge (t_n^\varepsilon + \varepsilon) \wedge T, \\ &u_n^\varepsilon := \mathcal{S}(t_{n+1}^\varepsilon - t_n^\varepsilon)\tilde{u}_n^\varepsilon, \quad \tilde{u}_{n+1}^\varepsilon := \mathcal{R}(t_{n+1}^\varepsilon, t_n^\varepsilon)u_n^\varepsilon;\end{aligned}$$

if $t_n^\varepsilon = T$, define $t_{n+1}^\varepsilon = T$ where $a \wedge b = \min\{a,b\}$. Then define the approximate solutions v^ε and $\tilde{v}^\varepsilon$ by

$$v^\varepsilon(t) := \mathcal{R}(t, t_n^\varepsilon)u_n^\varepsilon \quad \text{for } t \in [t_n^\varepsilon, t_{n+1}^\varepsilon) \quad \text{a.s.,} \tag{3.7}$$

$$\tilde{v}^\varepsilon(t) := \mathcal{S}(t - t_n^\varepsilon)\tilde{u}_n^\varepsilon \quad \text{for } t \in [t_n^\varepsilon, t_{n+1}^\varepsilon) \quad \text{a.s.} \tag{3.8}$$

We now derive the kinetic formulation satisfied by the approximate solutions v^ε, $\tilde{v}^\varepsilon$. Let $R > 0$ and let $\varphi \in C_c^\infty(D \times (-R,R))$. v^ε satisfies the strong kinetic formulation at every $t \in [t_n^\varepsilon, t_{n+1}^\varepsilon)$ by Lemma 3.1: P-a.s., for all $t \in [t_n^\varepsilon, t_{n+1}^\varepsilon)$,

$$\begin{aligned}&- \int_D \int_\mathbb{R} f^\pm(v^\varepsilon(t), \xi)\varphi \, d\xi dx + \int_D \int_\mathbb{R} f^\pm(u_n^\varepsilon, \xi)\varphi \, d\xi dx \\ &= - \sum_{k=1}^\infty \int_{t_n^\varepsilon}^t \int_D g_k(x, v^\varepsilon)\varphi(x, v^\varepsilon) \, dx d\beta_k(s) \\ &\quad - \frac{1}{2} \int_{t_n^\varepsilon}^t \int_D G^2(x, v^\varepsilon)\partial_\xi \varphi(x, v^\varepsilon) \, dx ds.\end{aligned} \tag{3.9}$$

On the other hand, note that $\tilde{v}^\varepsilon \in C([t_n^\varepsilon, t_{n+1}^\varepsilon); L^1(D))$ by Proposition 3.2. Hence $\tilde{v}^\varepsilon$ satisfies the strong kinetic formulation at every $t \in [t_n^\varepsilon, t_{n+1}^\varepsilon)$:

$$\begin{aligned}
&-\int_D \int_{\mathbb{R}} f^\pm(\tilde{v}^\varepsilon(t), \xi)\varphi \, d\xi dx + \int_D \int_{\mathbb{R}} f^\pm(\tilde{u}_n^\varepsilon, \xi)\varphi \, d\xi dx \\
&\quad + \int_{t_n^\varepsilon}^t \int_D \int_{\mathbb{R}} f^\pm(\tilde{v}^\varepsilon(s), \xi) a(\xi) \cdot \nabla\varphi \, d\xi dx ds \\
&\quad + M_R \int_{t_n^\varepsilon}^t \int_{\partial D} \int_{\mathbb{R}} f^\pm(u_b(s), \xi)\varphi \, d\xi d\sigma ds \\
&= \int_{D\times[t_n^\varepsilon, t]\times\mathbb{R}} \partial_\xi \varphi \, dm_n^\varepsilon + \int_{t_n^\varepsilon}^t \int_{\partial D} \int_{\mathbb{R}} \partial_\xi \varphi \, \bar{m}_{R,n}^{\pm,\varepsilon} \, d\xi d\sigma ds,
\end{aligned} \tag{3.10}$$

P-a.s., for all $t \in [t_n^\varepsilon, t_{n+1}^\varepsilon)$, where m_n^ε, $\bar{m}_{R,n}^{\pm,\varepsilon}$ are the associated entropy dissipation measures on $D\times[t_n^\varepsilon, t_{n+1}^\varepsilon)\times\mathbb{R}$ and $\partial D\times[t_n^\varepsilon, t_{n+1}^\varepsilon)\times\mathbb{R}$, a.s., respectively such that

$$\lim_{R\to\infty} m_n^\varepsilon(D \times [t_n^\varepsilon, t_{n+1}^\varepsilon) \times \{\xi \in \mathbb{R}; R \le |\xi|\}) = 0, \quad \text{a.s.,} \tag{3.11}$$

$$\lim_{\xi\uparrow R} \bar{m}_{R,n}^{+,\varepsilon}(x, t, \xi) = \lim_{\xi\downarrow -R} \bar{m}_{R,n}^{-,\varepsilon}(x, t, \xi) = 0, \quad \text{a.s.} \tag{3.12}$$

Therefore by (3.9) and (3.10) we have

$$\begin{aligned}
&-\int_D \int_{\mathbb{R}} f^\pm(v^\varepsilon(t), \xi)\varphi \, d\xi dx - \int_D \int_{\mathbb{R}} f^\pm(\tilde{v}^\varepsilon(t), \xi)\varphi \, d\xi dx \\
&\quad + \int_D \int_{\mathbb{R}} f^\pm(v^\varepsilon(t^\varepsilon), \xi)\varphi \, d\xi dx + \int_D \int_{\mathbb{R}} f^\pm(u_0, \xi)\varphi \, d\xi dx \\
&\quad + \int_0^t \int_D \int_{\mathbb{R}} f^\pm(\tilde{v}^\varepsilon(s), \xi) a(\xi) \cdot \nabla\varphi \, d\xi dx ds \\
&\quad + M_R \int_0^t \int_{\partial D} \int_{\mathbb{R}} f^\pm(u_b(s), \xi)\varphi \, d\xi d\sigma ds \\
&= -\sum_{k=1}^\infty \int_0^t \int_D g_k(x, v^\varepsilon)\varphi(x, v^\varepsilon) \, dx d\beta_k(s) \\
&\quad - \frac{1}{2} \int_0^t \int_D G^2(x, v^\varepsilon) \partial_\xi \varphi(x, v^\varepsilon) \, dx ds \\
&\quad + \int_{D\times[0,t]\times\mathbb{R}} \partial_\xi \varphi \, dm^\varepsilon + \int_0^t \int_{\partial D} \int_{\mathbb{R}} \partial_\xi \varphi \, \bar{m}_R^{\pm,\varepsilon} \, d\xi d\sigma ds,
\end{aligned} \tag{3.13}$$

a.s., for $\varphi \in C_c^\infty(D \times (-R,R))$ and $t \in [0,T^\varepsilon)$, where we have used the notations that $T^\varepsilon = \sup_{n\geq 1} t_n^\varepsilon$, $m^\varepsilon = \sum_{n=0}^\infty m_n^\varepsilon$ and $t^\varepsilon = t_k^\varepsilon$ if $t \in [t_k^\varepsilon, t_{k+1}^\varepsilon)$, $k \in \mathbb{N} \cup \{0\}$.

At the end of this section, we give some properties of the approximate solutions v^ε, $\tilde{v}^\varepsilon$, the measure m^ε and T^ε (for the proof see [16]).

Lemma 3.3. *For all $p \in [1,\infty)$ there exists a constant $C = C(p, u_0, u_b, T) \geq 0$ such that for all $\varepsilon \in (0,1)$, the solutions v^ε, $\tilde{v}^\varepsilon$ and the measure m^ε satisfy*

$$\mathbb{E} \sup_{t\in[0,T^\varepsilon)} \|v^\varepsilon(t)\|_{L^p(D)}^p \leq C, \quad \mathbb{E} \sup_{t\in[0,T^\varepsilon)} \|\tilde{v}^\varepsilon(t)\|_{L^p(D)}^p \leq C. \tag{3.14}$$

$$\mathbb{E}\left|m^\varepsilon(D \times [0,T^\varepsilon) \times \mathbb{R})\right|^2 \leq C, \tag{3.15}$$

and

$$\begin{aligned} \lim_{R\to\infty} \sup_{t\in[0,T^\varepsilon)} \mathbb{E} \int_D \{(v^\varepsilon(t) \mp R)^\pm\}^p \, dx = 0, \\ \lim_{R\to\infty} \sup_{t\in[0,T^\varepsilon)} \mathbb{E} \int_D \{(\tilde{v}^\varepsilon(t) \mp R)^\pm\}^p \, dx = 0. \end{aligned} \tag{3.16}$$

Moreover, for any $n \in \mathbb{N} \cup \{0\}$, $t_n^\varepsilon \leq s \leq t < t_{n+1}^\varepsilon$,

$$\mathbb{E}\|v^\varepsilon(t) - v^\varepsilon(s)\|_{L^1(D)} \leq CT\varepsilon^{1/2}. \tag{3.17}$$

$$\mathbb{E}\|\tilde{v}^\varepsilon(t) - \tilde{v}^\varepsilon(s)\|_{L^1(D)} \leq 2\varepsilon. \tag{3.18}$$

Proposition 3.4. *Let $\varepsilon > 0$. There exists a natural number $M = M(\varepsilon)$ such that $t_M^\varepsilon = T$.*

4 Outline of the proof of the main result

The uniqueness of kinetic solutions to (1.1)-(1.3) has been already obtained in [14, Corollary 1]. Consequently, we will give an outline of the proof for the existence of a kinetic solution under the hypotheses (H_1), (H_2) and (H_3).

We choose a finite open cover $\{U_{\lambda_i}\}_{i=0,\dots,L}$ of $\overline{D}$ and a partition of unity $\{\lambda_i\}_{i=0,\dots,L}$ on $\overline{D}$ subordinated to $\{U_{\lambda_i}\}$ such that $U_{\lambda_0} \cap \partial D = \emptyset$, for $i = 1,\dots,L$,

$$\begin{aligned} D_{\lambda_i} &:= D \cap U_{\lambda_i} = \{x \in U_{\lambda_i}; (\mathcal{A}_i x)_d > h_{\lambda_i}(\overline{\mathcal{A}_i x})\}, \\ \partial D_{\lambda_i} &:= \partial D \cap U_{\lambda_i} = \{x \in U_{\lambda_i}; (\mathcal{A}_i x)_d = h_{\lambda_i}(\overline{\mathcal{A}_i x})\}, \end{aligned}$$

with a Lipschitz function $h_{\lambda_i} : \mathbb{R}^{d-1} \to \mathbb{R}$, where $\mathcal{A}_i$ is an orthogonal matrix corresponding to a change of coordinates of $\mathbb{R}^d$ and $\bar{y}$ stands for $(y_1, \ldots, y_{d-1})$ if $y \in \mathbb{R}^d$. For the sake of clarity, we will drop the index i of λ_i and we will suppose that the matrix $\mathcal{A}_i$ equals the identity. We also set $Q_\lambda = D_\lambda \times (0,T)$, $\Sigma_\lambda = \partial D_\lambda \times (0,T)$ and $\Pi_\lambda = \{\bar{x}; x \in U_\lambda\}$.

To regularize functions that are defined on D_λ and $\mathbb{R}$, let us consider a standard mollifier ψ on $\mathbb{R}$, that is, ψ is a smooth, nonnegative and even function the support of which is in $(-1,1)$ such that $\int_{\mathbb{R}} \psi = 1$. We set $\rho^\lambda(x) = \Pi_{i=1}^{d-1} \psi(x_i)\psi(x_d - (L_\lambda + 1))$ for $x = (x_1, \ldots, x_d)$ with the Lipschitz constant L_λ of h_λ on Π^λ. For $\eta, \delta > 0$, we set $\rho^\lambda_\eta(x) = \frac{1}{\eta^d}\rho^\lambda(\frac{x}{\eta})$, $\psi_\delta(\xi) = \frac{1}{\delta}\psi(\frac{x}{\delta})$, $\alpha_{\eta,\delta} = \alpha_{\eta,\delta}(x,y,\xi,\zeta) = \rho^\lambda_\eta(y-x)\psi_\delta(\xi-\zeta)$, $\alpha^\lambda_{\eta,\delta} = \alpha_{\eta,\delta}\lambda(x)$. We also define the cutoff function as follows

$$\Psi_\kappa(\xi) = \int_{-\infty}^{\xi} \{\psi_\kappa(\zeta + R - \kappa) + \psi_\kappa(\zeta - R + \kappa)\}\, d\zeta,$$

for $\kappa > 0$. Set $\Psi_\kappa(\xi,\zeta) = \Psi_\kappa(\xi)\Psi_\kappa(\zeta)$.

Proposition 4.1 (Doubling of variables). *Let $\varepsilon, \varepsilon', \eta, \delta, R, \kappa > 0$ and set $B_R = (-R,R)$. Then for all $t \in [0,T)$ we have*

$$\begin{aligned}
&- \mathbb{E}\int_{D^2\times B_R^2} \Psi_\kappa(\xi,\zeta)\Big\{f^\pm(v^\varepsilon(x,t),\xi) + f^\pm(\tilde{v}^\varepsilon(x,t),\xi) - f^\pm(v^\varepsilon(x,t^\varepsilon),\xi)\Big\} \\
&\quad \times \Big\{f^\mp(v^{\varepsilon'}(y,t),\zeta) + f^\mp(\tilde{v}^{\varepsilon'}(y,t),\zeta) - f^\mp(v^{\varepsilon'}(y,t^{\varepsilon'}),\zeta)\Big\}\alpha^\lambda_{\eta,\delta}\, d\zeta d\xi dy dx \\
&\le \mathcal{E}^\pm(\kappa,\eta,\delta,\varepsilon,\varepsilon',\lambda,R),
\end{aligned}$$

where $\mathcal{E}^\pm(\kappa,\eta,\delta,\varepsilon,\varepsilon',\lambda,R)$ are non-negative functions which satisfy that for all positive null sequences $\{\varepsilon_n\}$ and $\{\varepsilon'_m\}$, there exist subsequences still denoted by $\{\varepsilon_n\}$ and $\{\varepsilon'_m\}$ such that

$$\liminf_{R\to\infty} \sum_{i=1}^{L} \lim_{\eta\downarrow 0} \lim_{\kappa\downarrow 0} \lim_{n,m\to\infty} \mathcal{E}^\pm(\kappa,\eta,\eta^{3/2},\varepsilon_n,\varepsilon'_m,\lambda_i,R) = 0. \tag{4.1}$$

We now proceed with the proof of the existence. By Proposition 4.1 it holds that for any $\varepsilon, \varepsilon', R, \eta, \delta, \kappa > 0$

$$\begin{aligned}
&\mathbb{E}\int_D (v^\varepsilon(x,t) - v^{\varepsilon'}(x,t))^\pm\, dx \le \left|\mathcal{I}^\pm(\varepsilon,\varepsilon',R)\right| \\
&\quad + \sum_{i=0}^{L}\sum_{k=1}^{4} \left|\mathcal{J}_k^\pm(\kappa,\eta,\delta,\varepsilon,\varepsilon',\lambda_i,R)\right| + \sum_{i=0}^{L} \mathcal{E}^\pm(\kappa,\eta,\delta,\varepsilon,\varepsilon',\lambda_i,R),
\end{aligned} \tag{4.2}$$

where

$$\mathcal{I}^{\pm}(\varepsilon,\varepsilon',R) = \mathbb{E}\int_D (v^{\varepsilon}(x,t) - v^{\varepsilon'}(x,t))^{\pm}\, dx \\ + \mathbb{E}\int_{D\times B_R} f_{\pm}(v^{\varepsilon}(x,t),\xi) f_{\mp}(v^{\varepsilon'}(x,t),\xi)\, d\xi dx,$$

$$\mathcal{J}_1^{\pm}(\kappa,\eta,\delta,\varepsilon,\varepsilon',\lambda,R) \\ = -\mathbb{E}\int_{D\times B_R} \lambda(x) f^{\pm}(v^{\varepsilon}(x,t),\xi) f^{\mp}(v^{\varepsilon'}(x,t),\xi)\, d\xi dx \\ + \mathbb{E}\int_{D^2\times B_R} \lambda(x) f^{\pm}(v^{\varepsilon}(x,t),\xi) f^{\mp}(v^{\varepsilon'}(y,t),\xi) \rho_{\eta}^{\lambda}(y-x)\, d\xi dy dx$$

$$\mathcal{J}_2^{\pm}(\kappa,\eta,\delta,\varepsilon,\varepsilon',\lambda,R) \\ = -\mathbb{E}\int_{D^2\times B_R} \lambda(x) f^{\pm}(v^{\varepsilon}(x,t),\xi) f^{\mp}(v^{\varepsilon'}(y,t),\xi) \rho_{\eta}^{\lambda}(y-x)\, d\xi dy dx \\ + \mathbb{E}\int_{D^2\times B_R^2} f^{\pm}(v^{\varepsilon}(x,t),\xi) f^{\mp}(v^{\varepsilon'}(y,t),\zeta) \alpha_{\eta,\delta}^{\lambda}\, d\zeta d\xi dy dx$$

$$\mathcal{J}_3^{\pm}(\kappa,\eta,\delta,\varepsilon,\varepsilon',\lambda,R) \\ = -\mathbb{E}\int_{D^2\times B_R^2} f^{\pm}(v^{\varepsilon}(x,t),\xi) f^{\mp}(v^{\varepsilon'}(y,t),\zeta) \alpha_{\eta,\delta}^{\lambda}\, d\zeta d\xi dy dx \\ + \mathbb{E}\int_{D^2\times B_R^2} \Psi_{\kappa}(\xi,\zeta) f^{\pm}(v^{\varepsilon}(x,t),\xi) f^{\mp}(v^{\varepsilon'}(y,t),\zeta) \alpha_{\eta,\delta}^{\lambda}\, d\zeta d\xi dy dx$$

$$\mathcal{J}_4^{\pm}(\kappa,\eta,\delta,\varepsilon,\varepsilon',\lambda,R) \\ = -\mathbb{E}\int_{D^2\times B_R^2} \Psi_{\kappa}(\xi,\zeta) f^{\pm}(v^{\varepsilon}(x,t),\xi) f^{\mp}(v^{\varepsilon'}(y,t),\zeta) \alpha_{\eta,\delta}^{\lambda}\, d\zeta d\xi dy dx \\ + \mathbb{E}\int_{D^2\times B_R^2} \Psi_{\kappa}(\xi,\zeta) \Big\{ f^{\pm}(v^{\varepsilon}(x,t),\xi) + f^{\pm}(\tilde{v}^{\varepsilon}(x,t),\xi) - f^{\pm}(v^{\varepsilon}(x,t^{\varepsilon}),\xi) \Big\} \\ \times \Big\{ f^{\mp}(v^{\varepsilon'}(y,t),\zeta) + f^{\mp}(\tilde{v}^{\varepsilon'}(y,t),\zeta) - f^{\mp}(v^{\varepsilon'}(y,t^{\varepsilon'}),\zeta) \Big\} \alpha_{\eta,\delta}^{\lambda}\, d\zeta d\xi dy dx$$

Moreover, $\mathcal{I}^{\pm}(\varepsilon,\varepsilon',R)$ is estimated as follows

$$
\begin{aligned}
&|\mathcal{I}^{+}(\varepsilon,\varepsilon',R)| \\
&\quad\le -\mathbb{E}\int_D\int_R^{\infty} f_+(v^{\varepsilon}(x,t),\xi)f_-(v^{\varepsilon'}(x,t),\xi)\,d\xi dx \\
&\qquad -\mathbb{E}\int_D\int_{-\infty}^{-R} f_+(v^{\varepsilon}(x,t),\xi)f_-(v^{\varepsilon'}(x,t),\xi)\,d\xi dx \\
&\quad\le \mathbb{E}\int_D\int_R^{\infty} f_+(v^{\varepsilon}(x,t),\xi)\,d\xi dx - \mathbb{E}\int_D\int_{-\infty}^{-R} f_-(v^{\varepsilon'}(x,t),\xi)\,d\xi dx \\
&\quad= \mathbb{E}\int_D (v^{\varepsilon}(x,t)-R)^{+}\,dx + \mathbb{E}\int_D (v^{\varepsilon'}(x,t)+R)^{-}\,dx.
\end{aligned}
\tag{4.3}
$$

Hence using (3.16), we have $\sup_{0<\varepsilon,\varepsilon'<1}|\mathcal{I}^{+}(\varepsilon,\varepsilon',R)| \to 0$ as $R\to\infty$. Similarly, $\sup_{0<\varepsilon,\varepsilon'<1}|\mathcal{I}^{-}(\varepsilon,\varepsilon',R)| \to 0$ as $R\to\infty$. We now show that there exist null sequences $\{\varepsilon_n\}$, $\{\varepsilon'_m\}$ such that

$$
\liminf_{R\to\infty}\sum_{i=1}^{L}\lim_{\eta\downarrow 0}\lim_{\kappa\downarrow 0}\lim_{n,m\to\infty}\mathcal{J}_k^{\pm}(\kappa,\eta,\eta^{3/2},\varepsilon,\varepsilon',\lambda_i,R)=0. \tag{4.4}
$$

By virtue of (3.18) we easily get $|\mathcal{J}_4^{\pm}(\kappa,\eta,\delta,\varepsilon,\varepsilon',\lambda,R)|\le C(\varepsilon+\varepsilon')$. Moreover, it is easy to see that $\sup_{0<\varepsilon,\varepsilon'<1}|\mathcal{J}_3^{\pm}(\kappa,\eta,\delta,\varepsilon,\varepsilon',\lambda,R)|\to 0$ as $\kappa\downarrow 0$. Next, $\mathcal{J}_2^{\pm}(\kappa,\eta,\delta,\varepsilon,\varepsilon',\lambda,R)$ is estimated as follows:

$$
\begin{aligned}
&|\mathcal{J}_2^{+}(\kappa,\eta,\delta,\varepsilon,\varepsilon',\lambda,R)| \\
&\quad\le \mathbb{E}\int_D\int_{\mathbb{R}}\int_{\mathbb{R}}\left|f^{-}(v^{\varepsilon'}(y,t),\xi)-f^{-}(v^{\varepsilon'}(y,t),\zeta)\right|\psi_\delta(\xi-\zeta)\,d\zeta d\xi dy \\
&\quad= \mathbb{E}\int_D\int_{\mathbb{R}}\psi(\zeta)\int_{\mathbb{R}}\left|f^{-}(v^{\varepsilon'}(y,t),\xi)-f^{-}(v^{\varepsilon'}(y,t),\zeta)\right|\,d\xi d\zeta dy \\
&\quad\le \mathbb{E}\int_D\int_{\mathbb{R}}|\delta\zeta|\psi(\zeta)\,d\zeta dy \le \delta|D|.
\end{aligned}
$$

We get a similar estimate for $\mathcal{J}_2^{-}(\kappa,\eta,\delta,\varepsilon,\varepsilon',\lambda,R)$. Finally $\mathcal{J}_1^{\pm}(\kappa,\eta,\delta,\varepsilon,\varepsilon',\lambda,R)$

is estimated as follows:

$$
\begin{aligned}
&|\mathcal{J}_1^+(\kappa,\eta,\delta,\varepsilon,\varepsilon',\lambda,R)| \\
&\quad\le \mathbb{E}\int_{D^2\times B_R}\left|f^-(v^{\varepsilon'}(y,t),\xi)-f^-(v^{\varepsilon'}(x,t),\xi)\right|\rho_\eta^\lambda(y-x)\;d\xi dy dx \\
&\quad= -\mathbb{E}\int_{D^2\times B_R} f^+(v^{\varepsilon'}(x,t),\xi)f^-(v^{\varepsilon'}(y,t),\xi)\rho_\eta^\lambda(y-x)\;d\xi dy dx \\
&\qquad -\mathbb{E}\int_{D^2\times B_R} f^-(v^{\varepsilon'}(x,t),\xi)f^+(v^{\varepsilon'}(y,t),\xi)\rho_\eta^\lambda(y-x)\;d\xi dy dx \\
&\quad\le \sum_{k=2}^{4}\mathcal{J}_k^+(\kappa,\eta,\delta,\varepsilon,\varepsilon',\lambda,R)+\sum_{k=2}^{4}\mathcal{J}_k^-(\kappa,\eta,\delta,\varepsilon,\varepsilon',\lambda,R) \\
&\qquad +\mathcal{E}^+(\kappa,\eta,\delta,\varepsilon,\varepsilon',\lambda,R)+\mathcal{E}^-(\kappa,\eta,\delta,\varepsilon,\varepsilon',\lambda,R).
\end{aligned}
$$

Thus we obtain the limit (4.4). Consequently, from (4.1), (4.2), (4.3) and (4.4) we have that $\{v^\varepsilon;\varepsilon>0\}$ is a Cauchy sequence in $L^\infty(0,T;L^1(\Omega\times D))$. Besides, by (3.17) and (3.18) we have

$$
\begin{aligned}
&\mathbb{E}\|v^\varepsilon(t)-\tilde{v}^\varepsilon(t)\|_{L^1(D)} \\
&\quad\le \mathbb{E}\|v^\varepsilon(t)-v^\varepsilon(t^\varepsilon)\|_{L^1(D)}+\mathbb{E}\|v^\varepsilon(t^\varepsilon)-\tilde{v}^\varepsilon(t)\|_{L^1(D)}\le C\varepsilon^{1/2}+\varepsilon
\end{aligned}
$$

for all $t\in[0,T)$. Therefore, $\{\tilde{v}^\varepsilon;\varepsilon>0\}$ is also a Cauchy sequence and it's limit is the same as the limit of $\{v^\varepsilon;\varepsilon>0\}$.

Once one has obtained that the approximate solution $\{v^\varepsilon\}$ (or $\{\tilde{v}^\varepsilon\}$) converges to u in the sense of $L^\infty(0,T;L^1(\Omega\times D))$-norm, one can proceed to the same arguments as in [4, Theorem 6.4]. In particular, $\{v^\varepsilon\}$ (or $\{v^{\varepsilon'}\}$) is a Cauchy sequence in $L^1(\Omega\times(0,T),\mathcal{P},dP\otimes dt;L^1(D))$, and hence the limit u is also predictable. From (3.15) and the definition of $\{\bar{f}^{\pm,\varepsilon}\}$ there a exist kinetic measure m and $\bar{f}^\pm\in L^\infty(\Sigma\times\mathbb{R})$ such that, up to subsequence,

$$
\begin{aligned}
m^{\varepsilon_n} &\rightharpoonup m \quad\text{in}\quad L^2_w(\Omega;\mathcal{M}_b)\text{-weak*}, \\
\bar{f}^{\pm,\varepsilon} &\rightharpoonup \bar{f}^\pm \quad\text{in}\quad L^\infty(\Omega\times\Sigma\times\mathbb{R})\text{-weak*},
\end{aligned}
$$

with $\mathbb{E}|m(D\times[0,T)\times\mathbb{R})|^2<\infty$. In particular m satisfies the decay condition (2.1). If we now set

$$
\bar{m}_R^+(x,t,\xi)=M_R(u_b(x,t)-\xi)^+-\int_\xi^R(-a(\zeta)\cdot\mathbf{n}(x))\bar{f}^+(x,t,\zeta)\;d\zeta,
$$

$$
\bar{m}_R^-(x,t,\xi)=M_R(u_b(x,t)-\xi)^--\int_{-R}^\xi(-a(\zeta)\cdot\mathbf{n}(x))\bar{f}^-(x,t,\zeta)\;d\zeta,
$$

we have, up to subsequence,

$$\bar{m}_R^{\pm,\varepsilon} \rightharpoonup \bar{m}_R^{\pm} \quad \text{in } \ L^\infty(\Omega \times \Sigma \times \mathbb{R})\text{-weak*},$$

and clearly $\bar{m}_R^{\pm}(x,t,\pm R \mp 0) = 0$ Let $\varphi \in C_c^\infty(D \times (-R,R))$. Then it is easy to see that

$$\int_D \int_{\mathbb{R}} f^+(v^{\varepsilon_n},\xi)\varphi(x,\xi)\ d\xi dx \to \int_D \int_{\mathbb{R}} f^+(u,\xi)\varphi(x,\xi)\ d\xi dx \quad \text{a.e. } \omega, t,$$

$$\begin{aligned} &\sum_{k=1}^{\infty} \int_0^t \int_D g_k(x,v^{\varepsilon_n})\varphi(x,v^{\varepsilon_n})\ dx d\beta_k(s) \\ &\qquad \to \sum_{k=1}^{\infty} \int_0^t \int_D g_k(x,u)\varphi(x,u)\ dx d\beta_k(s) \quad \text{in } \ L^2(\Omega), \end{aligned}$$

and

$$\begin{aligned} &\int_0^t \int_D G^2(x,v^{\varepsilon_n})\partial_\xi\varphi(x,v^{\varepsilon_n})\ dxds \\ &\qquad \to \int_0^t \int_D G^2(x,u)\partial_\xi\varphi(x,u)\ dxds, \quad \text{a.s.} \end{aligned}$$

Therefore passing to the limit in (3.13), we have

$$\begin{aligned} &- \int_D \int_{\mathbb{R}} f^\pm(u(t),\xi)\varphi\ d\xi dx + \int_D \int_{\mathbb{R}} f^\pm(u_0,\xi)\varphi\ d\xi dx \\ &\quad + \int_0^t \int_D \int_{\mathbb{R}} f^\pm(u(s),\xi)a(\xi)\cdot\nabla\varphi\ d\xi dx ds \\ &\quad + M_R \int_0^t \int_{\partial D} \int_{\mathbb{R}} f^\pm(u_b(s),\xi)\varphi(x,s,\xi) d\xi d\sigma ds \\ &= - \sum_{k=1}^{\infty} \int_0^t \int_D g_k(x,u(s))\varphi(x,u(s))\ dx d\beta_k(s) \\ &\quad - \frac{1}{2} \int_0^t \int_D G^2(x,u(s))\partial_\xi\varphi(x,u(s))\ dxds \\ &\quad + \int_{[0,t]\times D\times\mathbb{R}} \partial_\xi\varphi\ dm + \int_0^t \int_{\partial D} \int_{\mathbb{R}} \partial_\xi\varphi\, \bar{m}_R^{\pm}\ d\xi d\sigma ds, \end{aligned}$$

for a.e. ω, t. Multiplying the above by $\psi'(t)$, $\psi \in C_c^\infty([0,T))$, and integrating with respect to $t \in [0,T)$, we can see that u satisfies the kinetic formulation (2.3). Therefore we conclude that u is a kinetic solution to (1.1)-(1.3).

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